PROBABILITY THEORY

Joint probability, conditional probability and Bayes' theorem

Consider two events: A and B.

Let p(A) be the probability of the event A and p(B) the probability of event B.

The **joint probability** is p(A,B) or $p(A \cap B)$. That is p(A,B) is the probability that the two events will occur together. Obviously p(A,B) = p(B,A).

If A and B are independent the probability of both occurring is: p(A,B) = p(A)p(B).

If A and B are dependent the outcome or occurrence of the first affects the outcome or occurrence of the second, so $p(A,B) = p(A \mid B)p(B)$.

$p(A \mid B)$ is the the **conditional probability** of an event A in relationship to an event B. This is the probability that event A occurs given that event B has already occurred; in other words $p(A \mid B)$ is the probability of event A given B.

If the events are independent: $p(A \mid B) = p(A)$.

The probability that at least one of them will occur is given by:

$p(A \cup B) = p(A) + p(B) - p(A \cap B)$.

Bayes' theorem

Let p(A) and p(B) be the probabilities of A and B without regard to each other.

$p(A \mid B)$ is the conditional probability; that is the probability of observing event A given that B is true.

$p(B \mid A)$ is the probability of observing event B given that A is true.

So, $p(A \mid B)p(B) = p(B \mid A)p(A)$.

Example n. 1

Calculate the occurrence of an odd number when a die is rolled.

p(A) = 3/6 = 1/2.

Example n. 2

A coin is tossed and a die is rolled. Calculate the probability of landing on the head side of the coin and rolling a three on the die.

A = head on the coin. p(A) = 1/2.

B = three on the die. p(B) = 1/6.

The two events are independent. So, p(A,B) = p(A)p(B) = 1/2 * 1/6 = 1/12.

Example n. 3

A card is chosen at random from a deck of 52 cards. It is then replaced and a second card is chosen. What is the probability of choosing a jack and then a queen?

A = a jack on the first pick. p(A) = 4/52 = 1/13.

B = a queen on the second pick. p(B) = 4/52 = 1/13.

The two events are independent. So, p(A,B) = p(A)p(B) = 1/13*1/13 = 1/169.

Example n. 4

A card is chosen at random from a deck of 52 cards. Without replacing it, a second card is chosen. What is the probability of choosing an ace and then a king?

A = an ace on the first pick. p(A) = 4/52 = 1/13.

B = a king on the second pick.

p(B | A) = the probability of a king on 2^{nd} pick given an ace on 1^{st} pick = 4/51.

The two events are dependent. So, in this case p(A,B) = p(B | A)p(A) = 4/51*1/13 = 4/663.

Example n. 5

Three cards are chosen at random from a deck of 52 cards without replacement. What is the probability of choosing 3 aces?

A = an ace on the first pick. p(A) = 4/52.

B = an ace on the second pick. p(B | A) = 3/51.

C = an ace on the third pick. p(C | A,B) = 2/50.

The events are dependent.

p(A,B,C) = p(C | A,B)p(A,B) = p(C | A,B)p(B |A)p(A) = 2/50*3/51*4/52 = 25/132600 = 1/5525.

Example n. 6

In a school 70% of schoolboys like Chocolate, and 35% like Chocolate AND like Strawberry. What percent of those who like Chocolate also like Strawberry?

P(Strawberry|Chocolate) = P(Chocolate and Strawberry) / P(Chocolate) = 0.35 / 0.7 = 0.5

So 50% of schoolboys who like Chocolate also like Strawberry.

Example n. 7

Consider two urns. The urn A contains 3 white balls and 5 blue balls. The urn B contains 4 white balls and 4 blue balls. A coin is flipped: if it is Heads, a ball is drawn from the urn A, and if it is Tails, a ball is drawn from the urn B. What is the probability that a white ball was taken from the urn B?

Let A be the event that the coin flip was Head (the urn A is chosen), so $p(A) = 1/2$.

Let B be the event that the coin flip was Tails (the urn B is chosen), so $p(B) = 1/2$.

Let W be the event that a white ball is selected, in general.

From the given data, we know that:

$p(W \mid A) = 3/8$ (a white ball from urn A) and $p(W \mid B) = 4/8$ (a white ball from urn B).

We need to compute $p(B \mid W)$; so we can use the Bayes' Theorem.

$p(B \mid W) = p(B,W)/p(W)$

where $p(B,W) = p(W,B) = p(W \mid B)p(B) = 4/8*1/2 = 1/4$

and $p(W) = p(W,A)+p(W,B) = p(W \mid A)p(A) + p(W \mid B)p(B) = (3/8*1/2) + (4/8*1/2) = 3/16 + 1/4 = 7/16$

Finally, $p(B \mid W) = p(B,W)/p(W) = (1/4) / (7/16) = 16/28 = 4/7$.