A SHORT TREATISE ON MATRICES

In general terms, a matrix is a collection of data (numbers, symbols, or expressions) arranged in rows (horizontal) and columns (vertical).
Let A be a matrix with dimension m x n (m by n). Let a_{ij} be a generic element that occupies the position i,j (row i and column j). The A^{mxn} matrix can be indicated as follows:

$$\boldsymbol{A}=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & & & A_{2n} \\ \vdots & & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}$$

When the number of rows (m) is equal to the number of columns (n) the matrix is referred to as a **square matrix**.

A particular square matrix is the **identity matrix**, where all the elements are 0 except the principal diagonal whose items are 1. An example of identity matrix (3x3) is given below:

$$\boldsymbol{I}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For a square matrix is possible calculate a number called **determinant**, that can be denoted as det(A). It can be viewed as the scaling factor of the transformation described by the matrix.
If the rows and columns of the matrix are linearly independent from each other, the determinant will be a non-zero number.
Let’s see how to calculate a determinant.

First of all, let’s consider the simplest matrix: the **1x1 matrix** (that is made up of only one element).
For instance:
$A=(5)$
In this case the determinant is simply the element 5. That is, det(A) = 5.

For a **2x2 matrix** the determinant is the difference of the product of the elements of the two diagonals.
For instance:

$$A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$$

det(A) = (1x4) – (2x4) = 4 – 8 = -4
Therefore in general, the determinant of a 2x2 matrix is det(A) = (a_{11} a_{22}) – (a_{12} a_{21}).

For a **3x3 matrix** the calculation is more difficult. Let's see a numerical example:

$$A=\begin{pmatrix}1 & 2 & -1\\ 2 & -1 & 3\\ -2 & -1 & 4\end{pmatrix}$$

Let's consider the first row: (1 2 -1), and in particular the first element (a_{11}).
Let's consider the determinant of the 2x2 sub-matrix obtained from the original A matrix after cancelling out the row nr. 1 and the column nr. 1.
The sub-matrix is:
$\begin{pmatrix}-1 & 3\\ -1 & 4\end{pmatrix}$ and its determinant is (-1 x 4) – (3 x -1) = (-4) – (-3) = -1.
The determinant of the above sub-matrix is called **cofactor**, that can be denoted as C_{11}.
As regards the algebraic sign, for the generic cofactor C_{ij} the sign is $(-1)^{i+j}$.
In this case the sign is positive because $(-1)^{1+1} = (-1)^2 = +1$. Therefore C_{11} = -1.
For the second element (a_{12}), the cofactor C_{12} is:
determinant of $\begin{pmatrix}2 & 3\\ -2 & 4\end{pmatrix}$ = (2 x 4) – (3 x -2) = 8 – (-6) = 14.
In this case the sign of the cofactor is negative because $(-1)^{1+2} = (-1)^3 = -1$.
Therefore C_{12} = -14.
Finally for the third element (a_{13}) the cofactor is:
determinant of $\begin{pmatrix}2 & -1\\ -2 & -1\end{pmatrix}$ = (2 x -1) – (-1 x -2) = (-2) – (2) = -4.
In this case the algebraic sign of the cofactor is positive because $(-1)^{1+3} = (-1)^4 = +1$.
Therefore C_{13} = (+1)x(-4) = -4.
At this point, we can finally calculate the determinant of our original matrix A:
det(A) = $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ = 1x(-1) + 2x(-14) + (-1)x(-4) = (-1) + (-28) + (4) = -25.

The same result is obtained by selecting another row or a column.
For instance, if we consider the third column we have:

$\det(A) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

$C_{13} = \det\begin{pmatrix} 2 & -1 \\ -2 & -1 \end{pmatrix} = (-1)^{1+3}\,(2x(-1) – (-1)x(-2)) = +1\,(-2 – 2) = -4$

$C_{23} = \det\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} = (-1)^{2+3}\,(1x(-1) – 2x(-2)) = -1\,(-1 + 4) = -3$

$C_{33} = \det\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = (-1)^{3+3}\,(1x(-1) – 2x2) = +1\,(-1 – 4) = -5$

Therefore, det(A) = (-1)x(-4) + 3x(-3) + 4x(-5) = 4 – 9 – 20 = -25.

For a **3x3 matrix** another way to calculate its determinant is to apply the **rule of Sarrus*:***
by adding on the right the first and the second columns, the original matrix A becomes as follow:

$$\left[\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 2 \\ 2 & -1 & 3 & 2 & -1 \\ -2 & -1 & 4 & -2 & -1 \end{array}\right]$$

Now, the products of the elements of the red diagonals are taken with the positive sign, meanwhile the products of the elements of the green diagonals are taken with the negative sign.

$$\left[\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 2 \\ 2 & -1 & 3 & 2 & -1 \\ -2 & -1 & 4 & -2 & -1 \end{array}\right]$$

The determinant is calculated in this way:
det(A) = +(1x-1x4) +(2x3x-2) + (-1x2x-1) -(-1x-1x-2) -(1x3x-1) -(2x2x4) =
-4 + (-12) + 2 -(-2) -(-3) -(16) = -4 -12 +2 +2 +3 -16 = -25.